Linear Algebra
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11.1 - Change of Basis

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Coordinates of general vector space

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Definition

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Then the scalars $c_1, c_2, \ldots, c_n$ are called coordinates vector of v relative to the basis S.

The vector $\{c_1, c_2, \ldots, c_n\}$ in $\mathbb{R}^n$ is called the coordinates vector of v relative to the basis S, and is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

Remark.
A basis S of a vector space V is a set. This means that the order in which those vectors in S are listed does not generally matter.

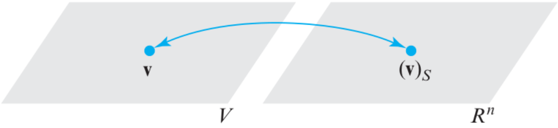
To deal with this, we define ordered basis, which is the basis in which the listing order of the basis vectors remains fixed.

Coordinates of general vector space

$\mathbf{v}_S$ is a vector in $\mathbb{R}^n$.

Once an ordered basis S is given for a vector space V, the "Uniqueness Theorem" establishes a one-to-one correspondence between vectors in V and vectors in $\mathbb{R}^n$.

A one-to-one correspondence



Example 1: coordinates relative to the standard basis for $\mathbb{R}^n$

For the vector space $V = \mathbb{R}^n$ and S is the standard basis, the coordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same;

$$\mathbf{v} = (\mathbf{v})_S$$

Example

For $V = \mathbb{R}^3$, $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

The representation of vector $\mathbf{v} = (a, b, c)$ in the standard basis is:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The coordinate vector relative to the basis S is $(\mathbf{v})_S = (a, b, c)$ (same as $\mathbf{v}$).

Example 2: coordinate vectors relative to standard bases

Find the coordinate vector for the polynomial:

$$\mathbf{p}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

relative to the standard basis for the vector space P_n.

Solution:

The standard basis for P_n is: $= \{1, x, x^2, \ldots, x^n\}$.

So, the coordinate vector for $\mathbf{p}$ relative to S is:

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n)$$

Example 3: coordinate vectors relative to standard bases

Find the coordinate vector of:

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

Solution:

The standard basis vectors for M_{22} is:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Hence,

$$B = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the coordinate vector of B relative to S is:

$$(B)_S = (a, b, c, d)$$

Exercise 1

Show that the following set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of $\mathbb{R}^3$.

$$\mathbf{v}_1 = (1, 2, 1), \ \mathbf{v}_2 = (2, 9, 0), \ \mathbf{v}_3 = (3, 3, 4)$$

Find the coordinate vector of $\mathbf{v} = (5, 1 - 9)$ relative to the basis S.

Solution: Question 1 (*skipped*)

Question 2:

We have to find the values c_1, c_2, c_3 s.t.:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or, in this case:

$$(5, 1 - 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

from which we can extract the linear equations system:

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \qquad\quad + 4c_3 = 9 \end{cases}$$

Solving the system, we obtain (verify it!):

$$c_1 = 1, \ c_2 = -1, \ c_3 = 2$$

This means that: $(\mathbf{v})_S = (1, -1, 2)$.

Exercise 2

Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ whose coordinate vector relative to $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = (1, 2, 1),\ \mathbf{v}_2 = (2, 9, 0),\ \mathbf{v}_3 = (3, 3, 4)$$

is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution:

Let: $(c_1, c_2, c_3) = (-1, 3, 2)$. Hence,

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) \\ &= (11, 31, 7) \end{aligned}$$

So, the vector $\mathbf{v}$ for which $(\mathbf{v})_S = (-1, 3, 2)$ is $(11, 31, 7)$.

Change of basis

Why change of basis needed?

- A basis that is suitable for one problem may not be suitable for another;
-

Coordinate maps

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a finite-dimensional vector space V. Let the coordinate vector of $\mathbf{v}$ relative to S be:

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

The one-to-one correspondence (mapping) between vectors in V and vectors in the Euclidean vector space $\mathbb{R}^n$ is defined as;

$$\mathbf{v} \rightarrow (\mathbf{v})_S$$

This is called the coordinate map relative to S from V to $\mathbb{R}^n$.

We will use column matrix to represent the coordinate vectors:

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The Change-of-Basis Problem

Problem: If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and we change the basis for V from a basis B to another basis B', how are the coordinate vector $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

- In the literature, B is usually called the old basis and B' is called the new basis.
- For convenience, I will use the terms **first basis** and **second basis**.

Solution of the Change-of-Basis problem (in 2-dimensional space)

Let

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \;\text{ and }\; B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$$

and the coordinate vectors for the 2nd basis relative to the 1st basis is:

$$[\mathbf{u}_1']_B = \begin{bmatrix} a \\ b \end{bmatrix} \;\text{ and }\; [\mathbf{u}_2']_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

i.e., the following relation holds:

$$\mathbf{u}_1' = a\mathbf{u}_1 + b\mathbf{u}_2 \tag{1}$$
$$\mathbf{u}_2' = c\mathbf{u}_1 + d\mathbf{u}_2 \tag{2}$$

Problem: Given a vector $\mathbf{v} \in V$, with

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

How to find the coordinate vector of $\mathbf{v}$ relative to B?

Solution (*cont.*)

Since the coordinate vector of $\mathbf{v}$ relative to B' is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

this means that:

$$\mathbf{v} = k_1\mathbf{u}_1' + k_2\mathbf{u}_2'$$

By the relation (1) and (2) in the previous slide, we have:

$$\begin{aligned} \mathbf{v} &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1 a + k_2 c)\mathbf{u}_1 + (k_1 b + k_2 b)\mathbf{u}_2 \end{aligned}$$

So, the coordinate vector of v relative to B is:

$$[\mathbf{v}]_B = \begin{bmatrix} k_1 + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$$

Finding transition matrices

The vector $[\mathbf{v}]_B = \begin{bmatrix} k_1 + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$ can be written as:

$$[\mathbf{v}]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{v}]_{B'}$$

Let $P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. This means that:

> *the coordinate vector $[\mathbf{v}]_B$ can be obtained by multiplying the coordinate vector $[\mathbf{v}]_{B'}$ on the left by matrix P.*

Solution of the Change-of-Basis Problem

Theorem

Let V be an n-dimensional space. If we want to change the basis for V from basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to another basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$.

Then for each vector $\mathbf{v} \in V$, we have the following relation between $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$, as follows:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$$

where P is the matrix whose columns are the coordinate vectors of B' relative to B, i.e., the columns of P are:

$$[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B$$

P is called the transition matrix from B' to B, and is denoted by $P_{B' \to B}$.

$$P_{B' \to B} = [\ [\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \ \ldots \ \mid [\mathbf{u}'_n]_B\ ] \quad (1)$$
$$P_{B \to B'} = [\ [\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \ \ldots \ \mid [\mathbf{u}_n]_{B'}\ ] \quad (2)$$

Example 1: finding transition matrices

Given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for $\mathbb{R}^2$, where:

$$\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (0, 1),\ \mathbf{u}_1' = (1, 1),\ \mathbf{u}_2' = (2, 1)$$

1. Find the transition matrix $P_{B' \to B}$ from B' to B.
2. Find the transition matrix $P_{B \to B'}$ from B to B'.

Solution of Example 1

Solution 1: The transition matrix $P_{B' \to B}$ from B' to B.

$$\mathbf{u}_1' = \mathbf{u}_1 + \mathbf{u}_2$$
$$\mathbf{u}_2' = 2\mathbf{u}_1 + \mathbf{u}_2$$

Hence,

$$[\mathbf{u}_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2']_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So,

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution of Example 1 (*cont.*)

Solution 2: The transition matrix $P_{B\to B'}$ from B to B'.

$$\begin{aligned}\mathbf{u}_1 &= -\mathbf{u}_1' + \mathbf{u}_2' \\ \mathbf{u}_2 &= 2\mathbf{u}_1 - \mathbf{u}_2\end{aligned}$$

Hence,

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix}-1\\1\end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{B'} = \begin{bmatrix}2\\-1\end{bmatrix}$$

So,

$$P_{B\to B'} = \begin{bmatrix}-1 & 2\\1 & -1\end{bmatrix}$$

Example 2: computing coordinate vectors

Problem:

Given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where:

$$\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (0, 1),\ \mathbf{u}'_1 = (1, 1),\ \mathbf{u}'_2 = (2, 1)$$

Find the vector $[\mathbf{v}]_B$ given that $[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

Solution:

$$[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Invertibility of transition matrices

What happen if we multiply $P_{B' \to B}$ with $P_{B \to B'}$?

- We first map the B-coordinates of $\mathbf{v}$ into its B'-coordinates;
- then map the B'-coordinates of $\mathbf{v}$ into its B-coordinates;
- This yields that $\mathbf{v}$ is back to its B-coordinates.

$$P_{B' \to B} P_{B \to B'} = P_{B \to B} = I$$

Example

Read again Example 1.

$$(P_{B' \to B})(P_{B \to B'}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem

$P_{B' \to B}$ is invertible, and its inverse is $P_{B \to B'}$.

A procedure for computing $P_{B \to B'}$

Procedure:

1. Form the matrix $[B' \mid B|]$;
2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form;
3. The resulting matrix will be $[I \mid P_{B \to B'}]$; *Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.*

Diagram:

$$\left[B' \mid B\right] \xrightarrow{\text{row operations}} \left[I \mid \text{transition from } B \text{ to } B'\right] \tag{1}$$

Exercise

In Example 1, we are given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where:

$$\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (0, 1),\ \mathbf{u}'_1 = (1, 1),\ \mathbf{u}'_2 = (2, 1)$$

Use formula (1) of the previous slide to find:

1. The transition matrix from B' to B.
2. The transition matrix from B to B'.

Solution of exercise

Question 1.

$$[B' \mid B] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Since the left side is already the identity matrix, no reduction is needed. Hence,

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Question 2.

$$[B' \mid B] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

By reducing the matrix, we obtain:

$$[I \mid \text{transition from } B \text{ to } B'] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Exercise (at home)

Given a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for $\mathbb{R}^2$, where:

$$\mathbf{u}_1 = (2, 1, 1),\ \mathbf{u}_2 = (2, -1, 1),\ \mathbf{u}_3 = (1, 2, 1)$$

$$\mathbf{u}'_1 = (3, 1, -5),\ \mathbf{u}'_2 = (1, 1, -3),\ \mathbf{u}'_3 = (-1, 0, 2)$$

1. Find the transition matrix from B to B'.
2. Find the transition matrix from the standard basis of $\mathbb{R}^3$ to B.
3. Find the transition matrix from the standard basis of $\mathbb{R}^3$ to B'.
4. Find the coordinate vector $\mathbf{w}$ relative to basis B, if the coordinate vector $\mathbf{w}$ relative to the standard basis S is $[\mathbf{w}]_S = (-5, 8, -5)$.

to be continued...